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The answer to Woodall's musquash problem

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Abstract

We prove that there are no n-agonal musquashes for n even with $n \neq 6$. This resolves a problem raised in Woodall's 1971 paper 'Thrackles and Deadlock'. © 1999 Elsevier Science B.V. All rights reserved.

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A classic problem in graph theory is to draw a graph in the plane with as few crossings as possible. By comparison, a *thrackle* is an unconventional concept, conceived some 40 years ago by John H. Conway; a thrackle is a drawing in which each pair of edges meets precisely once, either at a vertex or at a proper crossing (for recent work on thrackles, see [5]). In the late 1960s, the notion of a *musquash* appeared in Douglas R. Woodall's work on Conway's thrackle conjecture; a musquash is a thrackle of the n-gon with cyclic symmetry. Musquashes are known to exist for n odd and for $n = 6$, but do not exist for $n = 10$ nor for n a multiple of 4. Woodall's paper [6] concluded with the problem: "For which n, n even, does there exist an n-agonal musquash?" We prove

Theorem. *There are no n-agonal musquashes for n even with $n \neq 6$.*

To be precise, recall that an n-agonal musquash $\mathscr{M}$ is a planar n-gon, with $n \geqslant 5$, whose successive edges $e_1, \ldots, e_n$ are smooth curves without self-intersection such that:

(a) all intersections between the edges of $\mathscr{M}$ are normal (i.e., transverse) and occur outside the vertex set of $\mathscr{M}$,

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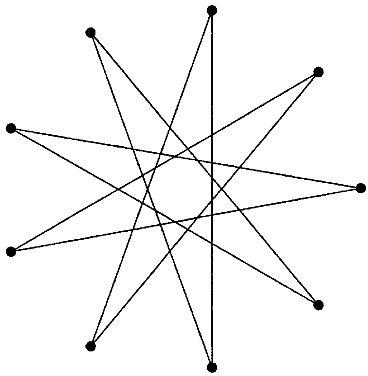
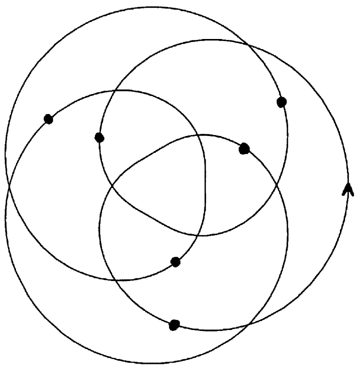


Fig. 1. Example of a 9-agonal musquash.



5	3	4
6	4	5
1	5	6
2	6	1
3	1	2
4	2	3

Fig. 2. A 6-agonal musquash.

(b) there are no intersections between any pair of consecutive edges,

(c) each edge intersects each of the remaining possible $n-3$ edges precisely once,

(d) if e_1 intersects edges in the following order: $e_{k_1},\dots,e_{k_{n-3}}$, then for all $i=2,\dots,n$, edge e_i intersects edges in the following order: $e_{k_1+i-1},\dots,e_{k_{n-3}+i-1}$, where the edge subscripts are computed modulo n.

An example of a 6-agonal musquash is given in Fig. 2 which also gives its intersection table: row i gives the sequence of edges met by e_i. Notice that the table is determined by its top row. Conversely, each permutation of the ordered set $\{3,\dots,n-1\}$ is the top row of a table corresponding to a drawing, on some surface, having the properties (a)–(d) above. For example, there is an n-agonal musquash for all n odd, given by the table with top row:

$$n-2, n-4, \dots, 5, 3, n-1, n-3, \dots, 6, 4.$$

In order to decide whether a given table is a musquash (i.e., can be realized by a curve *in the plane*), one can use Kuratowski's theorem. A quicker way is to use the table to write the 'Gauss word' of the associated curve and then use [3] or [1,2]. However, since there are $(n-3)!$ tables, neither method is a practical means of proving the existence or non-existence of musquashes for large n.

Proof of the Theorem. Suppose that $\mathscr{M}$ is an n-agonal musquash, where $n=2m$ and m is an odd integer. By stereographic projection, we lift $\mathscr{M}$ to the 2-sphere S^2. We begin by studying the symmetry of $\mathscr{M}$. First, for each $i=1,\dots,n$, let ϕ_i be the unique homeomorphism from e_i to e_{i+1} which preserves orientation, sends intersection points to intersection points, and with respect to the natural arc-length parameter, is an affine transformation on each of the connected components of the complement of the set of intersection points in e_i. Notice that because of condition (d) in the definition of a musquash, there is a well-defined homeomorphism ϕ of $\mathscr{M}$ such that $\phi_i=\phi|_{e_i}$ for each i. Obviously ϕ is periodic with period n.

Lemma 1. *ϕ extends to a homeomorphism Φ, of period n, of the sphere S^2.*

Proof. Notice that $\mathscr{M}$ forms the 1-skeleton of a 2-cell decomposition $\mathscr{S}$ of S^2. The faces of $\mathscr{S}$ are simply connected, and so each face F_i is homeomorphic to a regular planar polygon P_i; to fix ideas, use the regular polygons whose vertices are the kth roots of unity, for appropriate k. Moreover, choose the homeomorphisms $\psi_i\colon F_i \to P_i$ so that the end points and intersection points of each edge in $\mathscr{M}$ are sent to the vertices of P_i and such that the restriction of ψ_i^{-1} to each edge of P_i is an affine map. The map ϕ defines a bijection of the set of faces of $\mathscr{S}$, and if F_i is sent to F_j, then in fact $P_i=P_j$ and we can set $\Phi|_{F_i}=\psi_j^{-1}\circ\psi_i$. By construction, Φ agrees with ϕ on $\mathscr{M}$. □

Lemma 2. *Φ is orientation reversing.*

Proof. Consider the 2-cell decomposition $\mathscr{S}$ of S^2 determined by $\mathscr{M}$. Let V (resp. E, resp. F) denote the number of vertices (resp. edges, resp. faces) in $\mathscr{S}$. The Euler characteristic of S^2 is $V-E+F=2$. In the case at hand, $E=2V=2m(2m-3)$. Hence

$$F=2+m(2m-3). \tag{1}$$

Now assume that Φ is orientation preserving. By Eilenberg's theorem [4], Φ is conjugate to a rotation of order $n=2m$. In particular, Φ has exactly two fixed points. Consider the action of $\mathbb{Z}^n$ induced by Φ on the set of faces of $\mathscr{S}$. Notice that this action leaves two faces invariant and acts freely on the other faces; indeed, if for some integer i, the ith iterate Φ^i preserves a face, F_i say, then by Brouwer's fixed point theorem, F_i contains a fixed point of Φ^i, and hence either $\Phi^i=\mathrm{id}$ or F_i contains a fixed point of Φ. It follows from the stabilizer-orbit theorem that

$$F=2+2mk, \tag{2}$$

for some integer k. Finally notice that (1) and (2) are contradictory, since (1) implies that F is odd, as m is odd, while (2) implies that F is even. □

Recall that the intersections on the edges e_i can be given a positive or negative sign as follows: if e_i intersects e_j, then this is a positive (resp. negative) intersection if as

Table 1

−5	3	4
6	−4	−5
−1	5	6
2	−6	−1
−3	1	2
4	−2	−3

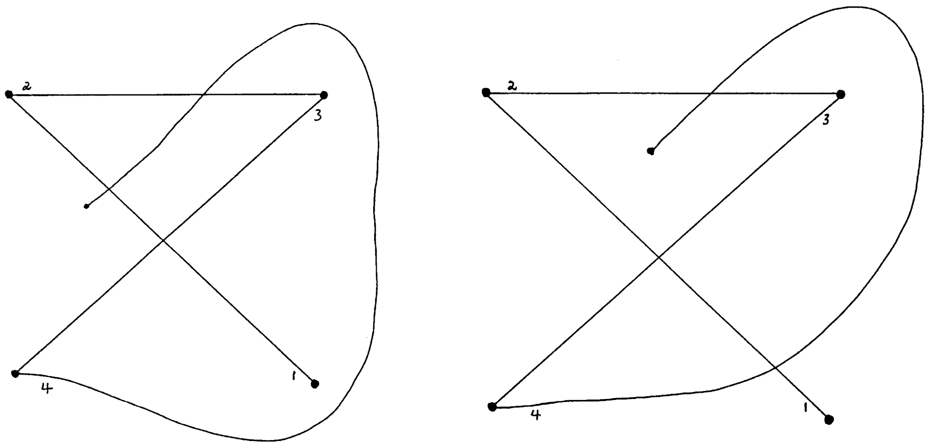
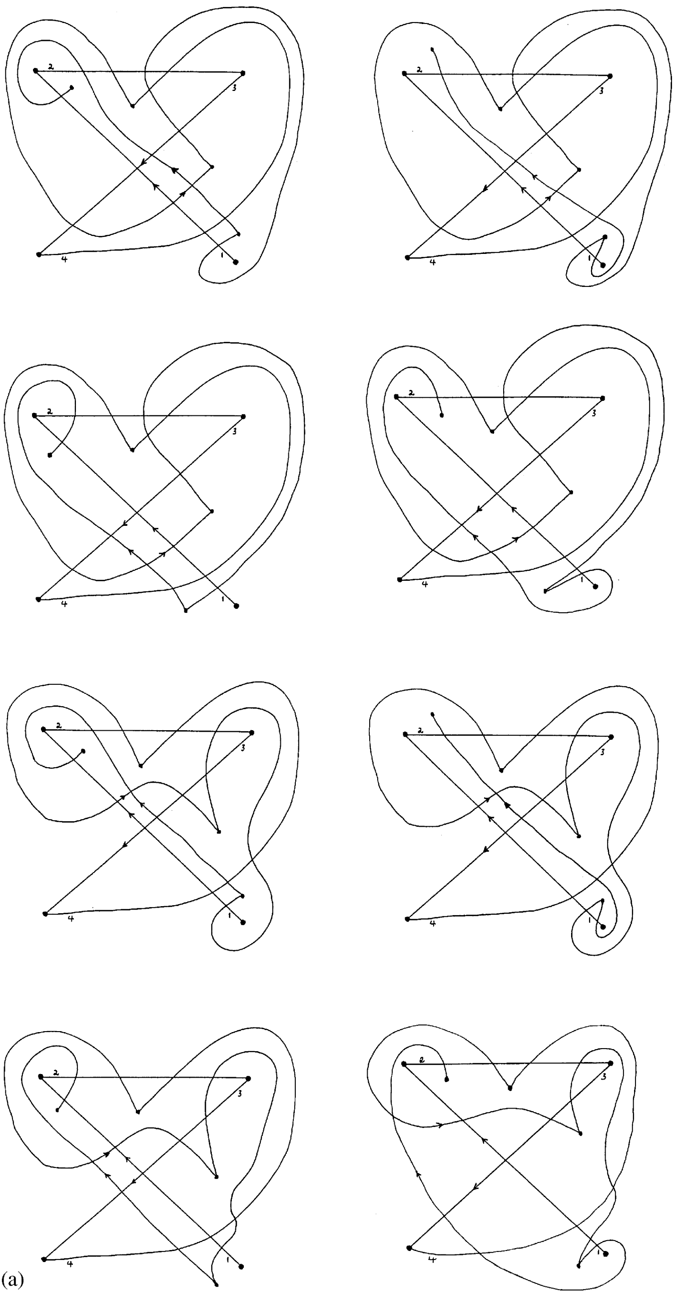


Fig. 3. $-3 \prec -4$ and $4 \prec -3$.

one passes along e_i, the curve e_j crosses e_i from left to right (resp. right to left). For example, Table 1 gives the intersection table with signs, for the musquash of Fig. 2. Notice that for an arbitrary musquash $\mathscr{M}$, with n even, Lemma 2 implies that the signs alternate down each of the columns of its intersection table.

We now investigate what happens when we try to draw the musquash $\mathscr{M}$ edge by edge, starting with e_1. Let $\mathscr{M}_k$ denote the drawing of the first k edges of $\mathscr{M}$. We will consider all the possibilities for $\mathscr{M}_k$ for $k = 1, 2, 3, \ldots, 7$. We do this sequentially; for a possible drawing of $\mathscr{M}_i$, we consider all the possibilities for $\mathscr{M}_{i+1}$ that are allowed when e_{i+1} is drawn. The first consideration occurs with the intersection between e_3 and e_1 in M_3. The only choice here is that in row 1 of the intersection table, one could have $+3$ or -3. In fact, by applying the antipodal map of S^2 if necessary, we may restrict our attention to the -3 case. When we draw in e_4 there are, a priori, 4 possibilities: $-3 \prec 4$, $-3 \prec -4$, $4 \prec -3$ or $-4 \prec -3$, where $a \prec b$ means that a precedes b in row 1. However, as the intersection of e_1 with e_3 is negative, the intersection e_2 with e_4 must be positive, by the previous paragraph. It follows that the cases $-3 \prec 4$ and $-4 \prec -3$ are impossible. Drawings of the possible cases $-3 \prec -4$ and $4 \prec -3$ are given in Fig. 3. Notice that if $\mathscr{M}$ has $-3 \prec -4$, then retracing $\mathscr{M}$ in the opposite direction, starting at the end of edge e_4 of $\mathscr{M}$, one obtains a new musquash $\mathscr{M}'$ with $4 \prec -3$. So it suffices to consider the case $4 \prec -3$. Now consider all possible ways of drawing in edges e_5, e_6, e_7, assuming $4 \prec -3$. One easily finds that there are only 3



(a)

Fig. 4.

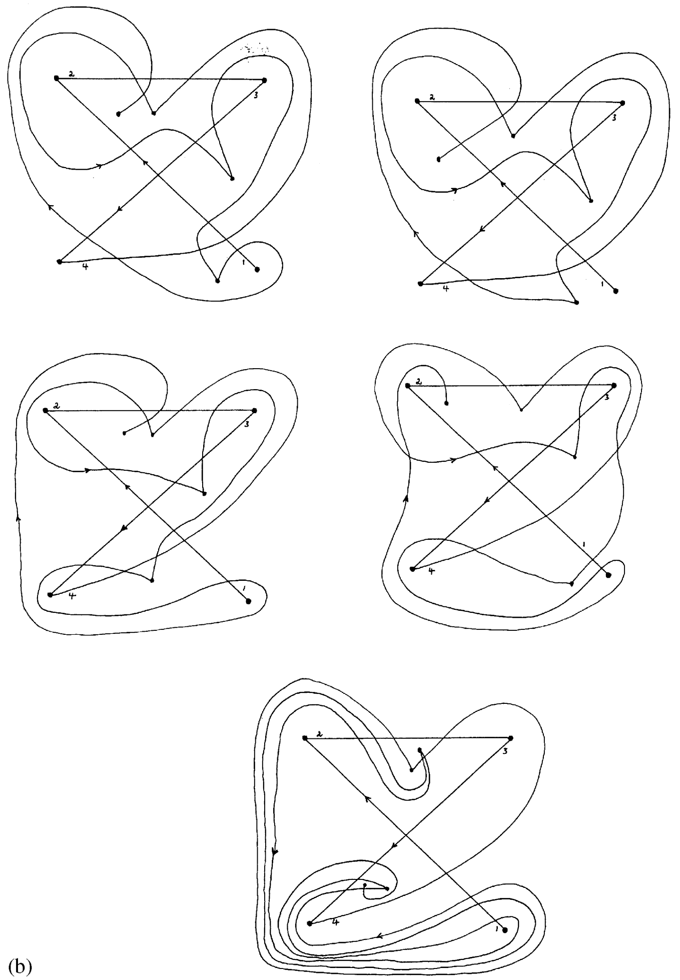


Fig. 4. (Contd.).

ways of drawing e_5 (bearing in mind that one is working on S^2 rather than the plane), 6 ways of drawing e_6 and altogether, 13 ways of drawing e_7; a sketch of each of the 13 cases is given in Fig. 4.

Returning to the symmetry Φ, consider the homeomorphism $\Phi^2 = \Phi \circ \Phi$. By Eilenberg's theorem, Φ^2 is conjugate to a rotation of order m. In particular, Φ^2 has exactly two fixed points, x_1 and x_2 say.

Notice that Φ interchanges the points x_1 and x_2. Indeed, $\Phi(x_1)$ and $\Phi(x_2)$ are fixed points of Φ^2 and so Φ either interchanges x_1 and x_2, or Φ fixes x_1 and x_2. In the latter case, since the fixed point set of Φ is a subset of the fixed point set of Φ^2, x_1 and x_2 would be the only fixed points of Φ, and in particular, Φ would have isolated fixed points, which is impossible for an orientation reversing homeomorphism of S^2.

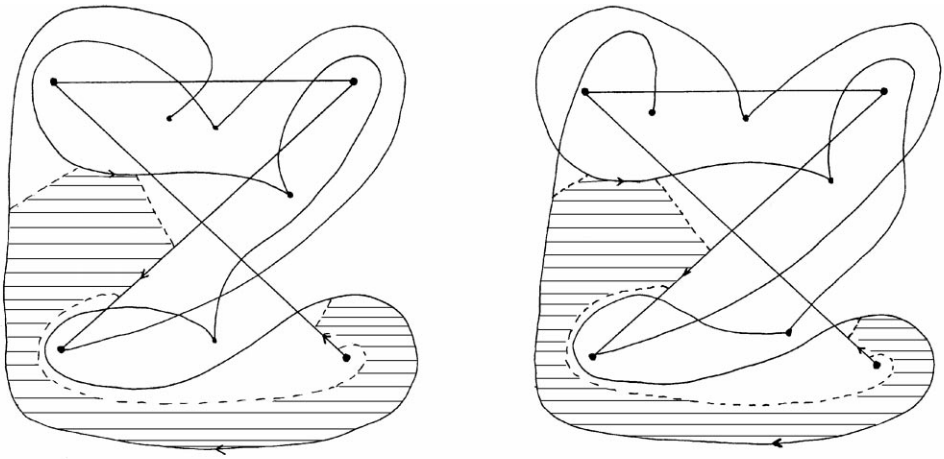


Fig. 5.

Consider the faces F_1 and F_2 of $\mathscr{S}$ containing x_1 and x_2, respectively. From the above, Φ interchanges F_1 and F_2, while Φ^2 preserves both F_1 and F_2. Notice that if an edge e_i is incident with the boundary of F_1, then from the definition of Φ, the edges e_{i+k} are all incident with the boundary of F_1 (resp. F_2), for k even (resp. for k odd), where the edge subscripts are computed modulo n. In particular, e_1 is necessarily incident with the boundary of one of the faces, say F_1, and thus e_k is incident with the boundary of F_1 for all k odd. So, as one travels around the boundary of F_1, one encounters segments of the edges e_1, e_3, e_5, etc., but not necessarily in this order. Moreover, and this is the decisive point, since Φ preserves the orientation of $\mathscr{M}$, the edges e_1, e_3, e_5, e_7, all have the same orientation in the boundary of F_1.

Now consider the diagrams in Fig. 4. From the above considerations, if one of these diagrams could be completed to form an n-agonal musquash, then in that diagram the face F_1 must appear as a polygonal region whose boundary contains segments of the edges e_1, e_3, e_5, e_7 with the same orientation. Cutting off some of the corners of F_1, we obtain an octagon $\mathscr{O}$, 4 of whose edges are segments of the edges e_1, e_3, e_5, e_7 with the same orientation, and $\mathscr{O}$ has no other contact with $\mathscr{M}$. For each diagram in Fig. 4, there is a small finite number of ways of drawing an octagon such that 4 of its edges are segments of e_1, e_3, e_5, e_7. However, one easily verifies that there are only two cases where it is possible to draw an octagon such that the edges e_1, e_3, e_5, e_7 all have the same orientation; they are shown in Fig. 5. There are several ways to eliminate these final two cases. Perhaps the easiest way is to examine the face F_2 containing the other fixed point x_2. Notice that in each of the cases at hand, there is a unique way to draw in e_8 (see Fig. 6), and it is easy to see that no face contains an octagon for which four of its edges are segments of the edges e_2, e_4, e_6, e_8 with the same orientation. This completes the proof of the theorem.

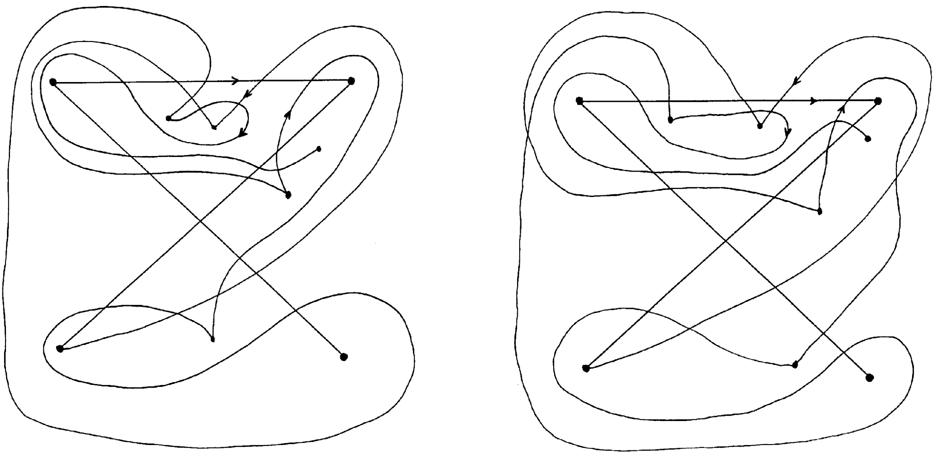


Fig. 6.

Acknowledgements

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